4

Calculus of Variations

Structure

4.1. Introduction. This chapter contains methods to obtain extremum of a given functional in one variable, several variable, for functional involving higher derivatives and variational derivatives.

4.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Brachistochrone problem.
(ii) Geodesics Problem.
(iii) Isoperimetric Problem.
(iv) The problem of minimum surface of revolution.

4.1.2. Keywords. Fumctional, extremal, Euler Equation.

4.2. Functional. Let there be a class of functions. By a functional, we mean a correspondence which assigns a definite real number to each function belonging to the class. In other words, a functional is a kind of function where the independent variable is itself a function. Thus, the domain of the functional is the set of functions.

Examples. (1) Let y(x) be an arbitrary continuously differentiable function defined on interval [a, b]. Then the formula,

$J[y] = \int_a^b y'^2(x)\,dx$ defines a functional on the set of all such functions y(x).

(2) Consider the set of all rectifiable plane curves. The length of any curve between the points (x_0, y_0) and (x_1, y_1) on the curve y = y(x) is given by

$$l\big(y(x)\big) = \int_{x_0}^{x_1} \left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$$

This defines a functional on the set of rectifiable curves.

(3) Consider all possible paths joining two given points A and B in the plane. Suppose that a particle can move along any of these paths and let the particle have a definite velocity v (x, y) at the point (x, y). Then we can define a functional by associating with each path, the time taken by the particle to traverse the path.

(4) The expression,

$$J\,[y] = \int_a^b F[x\,,\, y(x)\,,\, y'(x)]dx$$

gives a general example of functional. By choosing different functions F (x, y, y′) we obtain different functions. For example, if we take $F \equiv y'^2$, we obtain example (1) given above.

Remark. The generalised study of functionals is called "calculus of functionals". The most developed branch of calculus of functionals is concerned with finding the maxima and minima of functionals and is called "calculus of variation". We shall study this particular branch and shall find the extremals of the functionals.

Further, while finding the extremals we shall consider only the functionals of the type

$$J\,[y] = \int_a^b F[x\,,\, y(x)\,,\, y'(x)]dx$$

where y(x) ranges over the set of all continuously differentiable functions defined on the interval [a, b].

4.2.1. Motivating Problems. The following are some problems involving the determination of maxima and minima of functionals. These problems motivated the development of the subject.

1. **Brachistochrone problem.** Let A and B be two fixed points. Then the time taken by a particle to slide under the influence of gravity along some path joining A and B depends on the choice of the path (curve) and hence is a functional. The curve such that the particle takes the least time to go from A to B is called "brachistochrone".

The brachistochrone problem was posed by John Bernoulli in 1696 and played an important part in the development of the calculus of variation. The problem was solved by John Bernoulli, James Bernoulli, Newton and L Hospital. the brachistocharone comes out to be a cycloid lying in the vertical plane and passing through A and B.

2. **Geodesics Problem.** In this problem, we have to determine the line of shortest length connecting two given points (x_0, y_0, z_0) and (x_1, y_1, z_1) on a surface S given by ϕ (x, y, z) = 0.

Mathematically, we are required to minimize the arc length *l* joining the two points on S given by

$$l = \int_{x_0}^{x_1} \left[1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2\right]^{1/2} dx$$

Subject to the constraint $\phi(x, y, z) = 0$

3. Isoperimetric Problem. In this problem, we have to find the extremal of a functional under the constraint that another functional assumes a constant value.

Mathematically, to make

$$J[y] = \int_{x_0}^{x_1} F[x, y(x), y'(x)]dx$$

maximum or minimum such that the functional

$$\phi[y] = \int_{x_0}^{x_1} G[x, y(x), y'(x)]dx \text{ is kept constant.}$$

For example, "Among all closed curves of a given length l, find the curve enclosing the greatest area." This problem is an isoperimetric problem. This was solved by Euler and required curve comes out to be a circle.

4. **The problem of minimum surface of revolution.** In this problem, we have to find a curve y = y(x) passing through two given points (x_0, y_0) and (x_1, y_1) which when rotated about the x – axis gives a minimum surface area.

Mathematically, the surface area bounded by such curve is given by,

$$S = \int_{x_0}^{x_1} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Thus, we have to find a particular curve y = y(x) which minimizes S.

4.2.2. Function Spaces. As in study of functions, we use geometric language by taking a set of n numbers (x_1, x_2, …, x_n) as a point in an n – dimensional space. In the same way in the study of functionals, we shall regard each functions y(x) belonging to some class as a point in some space and the spaces whose elements are functions are called "Function spaces".

Remark. The concept of continuity plays an important role for functionals, just as it does for the ordinary functions. In order to formulate this concept for functionals, we must somehow introduce a concept of closeness for elements in a function space. This is most conveniently done by introducing the concept of the "norm" of a function (analogous to the concept of distance in Euclidean space). For this, we introduce the following basic concepts starting with a linear space.

4.2.3. Linear space. By a linear space, we mean a set $\mathbb{R}$ of elements x, y, z, … of any kind for which the operations of addition and multiplication by real numbers $\alpha, \beta, \ldots$ are defined and obey the following axioms:

i. x + y = y + x

ii. (x + y) + z = x + (y + z)

iii. There exists an element '0' such that x + 0 = x = 0 + x for all $x \in \mathbb{R}$

iv. For each $x \in \mathbb{R}$, there exists an element –x s.t. x + (–x) = 0 = (–x) + x

v. 1.x = x

vi. $\alpha(\beta x) = (\alpha\beta)x$

vii. $(\alpha + \beta)x = \alpha x + \beta x$

viii. $\alpha(x + y) = \alpha x + \alpha y$

4.2.4. Normed Linear space. A linear space $\mathbb{R}$ is said to be normed if each element $x \in \mathbb{R}$ is assigned a non negative number $\|x\|$, called the norm of x, such that

(1) $\|x\| = 0$ if and only if x = 0

(2) $\|\alpha x\| = |\alpha|\ \|x\|$

(3) $\|x + y\| \le \|x\| + \|y\|$

In a normed linear space, we can talk about distances between elements by defining the distance between x and y to be the quantity $\|x - y\|$.

4.2.5. Important Normed Linear spaces. Here are some examples of normed linear spaces which will be commonly used in our further study.

(1) The space C [a, b] : The space consisting of all continuous functions y(x) defined on a closed interval [a, b] is denoted as C [a, b]. By addition of elements of C [a, b] and multiplication of elements of C[a, b] by numbers, we mean ordinary addition of functions and multiplication of functions by numbers.

The norm in C [a, b] is defined as:

$$\|y\|_0 = \max_{a \le x \le b} . | y(x) |$$

(2) The space D [a, b] : This space consists of all functions y(x) defined on an interval [a, b] which are continuous and have continuous first derivatives. The operations of addition and multiplication by numbers are the same as in C [a, b] but the norm is defined by the formula,

$$\|y\|_1 = \max.\ a \le x \le b\ |y(x)| + \max.\ a \le x \le b\ |y'(x)|$$

Thus, two functions in D [a, b] are regarded as close together if both the functions themselves and their first derivatives are close together, since

$$\|y - z\|_1 < \in \text{ implies that } |y(x) - z(x)| < \in \text{ and } |y'(x) - z'(x)| < \in .$$

(3) The Space D_n [a, b] : The space D_n [a, b] consists of all functions y(x) defined on an interval [a, b] which are continuous and have continuous derivatives upto order n where n is fixed integer. Addition of elements of D_n and multiplication of elements of D_n by numbers are defined just as in the preceding cases, but the norm is defined as :

$$\|y\|_n = \sum_{i=0}^{n} \max.\ a \le x \le b \left|y^{(i)}(x)\right|$$

where $y^{(i)}(x) = \left(\frac{d}{dx}\right)^i$ y(x) and $y^{(0)}(x)$ denotes the function y(x) itself.

Thus two functions in D_n [a, b] are regarded as close together if the values of the functions themselves and of all their derivatives upto order n inclusive are close together.

4.2.6. Closeness of functions.

(1) The functions y(x) and z(x) are said to be close in the sense of zero order proximity if the value $|y(x) - z(x)|$ is small for all x for which the functions are defined. Thus, in the space C [a, b], the closeness is in the sense of zero order proximity.

(2) The functions y(x) and z(x) are said to be close in the sense of first order proximity if both $|y(x) - z(x)|$ and $|y'(x) - z'(x)|$ are small for all values of x for which the functions are defined. Thus, in the space D [a, b], the closeness is in the sense of first order proximity.

(3) The functions y(x) and z(x) are said to be close in the sense of n^{th} order proximity if

$|y(x) - y(x)|$, $|y'(x) - z'(x)|, \ldots,$ $|y^n(x) - z^n(x)|$ are small for all values of x for which the functions are defined. In the space D_n [a, b], the closeness is in the sense of n^{th} order proximity.

4.2.7. Continuity of functional.

The functional J [y(x)] is said to be continuous at the point y = y_0(x) if for any $\in$ > 0, there exists a δ > 0 such that

$$|J[y] - J[y_0]| < \in \text{ whenever } \|y - y_0\| < \delta$$

Further, the continuity is said to be in the sense of zero, first or n^{th} order proximity according as norm is defined as in C [a, b], D [a, b] or D_n [a, b].

4.2.8. Linear functional : Given a normed linear space $\mathbb{R}$, let each element h $\in \mathbb{R}$ be assigned a number ϕ [h] that is, let ϕ [h] be a functional defined on $\mathbb{R}$. Then ϕ [h] is said to be a linear functional if

(i) $\phi[\alpha h] = \alpha \phi[h]$ for any h $\in \mathbb{R}$ and any real number α.

(ii) $\phi[h_1 + h_2] = \phi[h_1] + \phi[h_2]$ for any $h_1, h_2 \in \mathbb{R}$.

For example,

(1) If we associate with each function h(x) $\in$ C [a, b] its value at a fixed point x_0 in [a, b] that is, if we define the functional ϕ[h] by the formula $\phi[h] = h(x_0)$ then ϕ[h] is a linear functional on C [a, b].

(2) The integral $\phi[h] = \int_a^b h(x)\, dx$ defines a linear functional on C [a, b].

(3) The integral $\phi[h] = \int_a^b \alpha(x)\, h(x)\, dx$ where α(x) is a fixed function in C [a, b] defines a linear functional on C [a, b].

(4) More generally, the integral

$$\phi[h] = \int_a^b \left[\alpha_0(x)\, h(x) + \alpha_1(x)\, h^1(x) + ... + \alpha_n(x)\, h^n(x)\right] dx$$

where the α_i(x) are fixed functions in C [a, b] defines a linear functional on D_n [a, b]

4.2.9. Lemma. If α(x) is continuous in [a, b] and if $\int_a^b \alpha(x)\, h(x)\, dx = 0$ for every function h(x) $\in$ C [a, b] such that h(a) = h(b) = 0, then α(x) = 0 for all x $\in$ [a, b]

Proof. Let, if possible, the function α(x) be non – zero say positive, at some point of [a, b]. Then by virtue of continuity, α(x) is also positive in some interval $[x_1, x_2] \subseteq$ [a, b].

We set,

$$h(x) = \begin{cases} (x-x_1)(x_2-x), & \text{for all } x \in [x_1, x_2] \\ 0, & \text{otherwise} \end{cases}$$

Then, h(x) obviously satisfies the conditions of the lemma. But we have

$$\int_a^b \alpha(x)\, h(x)\, dx = \int_{x_1}^{x_2} \alpha(x)(x-x_1)\,(x_2-x)\, dx > 0,$$

since the integrand is positive (except at x_1 and x_2). This contradiction proves the lemma.

Remark. The lemma still holds if we replace C [a, b] by D_n [a, b].

4.2.10. Lemma. If α (x) is continuous in [a, b] and if $\int_a^b \alpha(x)\, h'(x)\, dx = 0$, for every h(x) in D_1[a, b] such that h(a) = h(b) = 0, then α (x) = C for all x $\in$ [a, b] where C is a constant.

Proof. Let C be the constant defined by the condition

$$\int_a^b \left[\alpha(x) - C\right] dx = 0 \qquad (1)$$

which, in fact, gives C = $\frac{1}{b-a} \int_a^b \alpha(x)\, dx$

Also, let h(x) = $\int_a^b \left[\alpha(\xi) - C\right] d\xi$ then clearly h(x) $\in$ D_1 [a, b]. Also we have,

$$h(a) = \int_a^b \left[\alpha(\xi) - C\right] d\xi = 0 \text{ and } h(b) = \int_a^b \left[\alpha(\xi) - C\right] d\xi = 0 \qquad \text{[By (1)]}$$

Thus h(x) satisfies all the conditions of the lemma and so by given hypothesis.

$$\int_a^b \alpha(x)\, h'(x)\, dx = 0 \qquad (2)$$

Now we calculate

$$\int_a^b \left[\alpha(x) - C\right] h'(x)\, dx = \int_a^b \alpha(x)\, h'(x)\, dx - C \int_a^b h'(x)\, dx = 0 - C \int_a^b h'(x)\, dx \qquad \text{[By (2)]}$$

$$= -C\left[h(b) - h(a)\right] = -C\,[0 - 0] = 0 \qquad (3)$$

On the other hand, by definition of h(x)

$$\int_a^b [\alpha(x) - C]\ h^1(x)\ dx = \int_a^b [\alpha(x) - C][\alpha(x) - C]\ dx = \int_a^b [\alpha(x) - C]^2\ dx \qquad (4)$$

Expression (3) and (4) give the value of same integral so we have

$$\int_a^b [\alpha(x) - C]^2\ dx = 0 \Rightarrow \alpha(x) - C = 0 \qquad \Rightarrow \alpha(x) = C \text{ for all } x \in [a, b]$$

4.2.11. Lemma. If α (x) is continuous in [a, b] and if $\int_a^b \alpha(x)\ h''(x)\ dx = 0$, for every function h(x) $\in$ D_2 [a, b] such that h(a) = h(b) = 0 and h'(a) = h'(b) = 0. Then α (x) = C_0 + C_1 x for all x $\in$ [a, b] where C_0 and C_1 are constants.

Proof. Let C_0 and C_1 be defined by the conditions

$$\int_a^b [\alpha(x) - C_0 - C_1\ x]dx = 0 \qquad (1)$$

$$\int_a^b \int_a^x [\alpha(\xi) - C_0 - C_1\ \xi]d\xi\ dx = 0 \qquad (2)$$

and let $h(x) = \int_a^x \int_a^\xi [\alpha(t) - C_0 - C_1\ t]dt\ d\xi \qquad (3)$

$$\Rightarrow \qquad h'(x) = \int_a^x [\alpha(t) - C_0 - C_1\ t]dt \qquad (4)$$

and $\qquad h''(x) = \alpha(x) - C_0 - C_1x \qquad (5)$

Then, clearly h(x) $\in$ D_2[a, b].

Also, we have

$$h(a) = \int_a^a \int_a^\xi [\alpha(t) - C_0 - C_1\ t]dt\ d\xi = 0$$

$$h(b) = \int_a^b \int_a^\xi [\alpha(t) - C_0 - C_1\ t]dt\ d\xi = 0 \qquad \text{[By (2)]}$$

$$h'(a) = \int_a^a [\alpha(t) - C_0 - C_1\ t]dt = 0 \text{ and } h'(b) = \int_a^b [\alpha(t) - C_0 - C_1\ t]dt = 0 \text{ [By (1)]}$$

Thus h(x) satisfies all the conditions of the lemma and so by given hypothesis,

$$\int_a^b \alpha(x)\, h''(x)\, dx = 0 \qquad (6)$$

Now we calculate,

$$\int_a^b \left[\alpha(x) - C_0 - C_1 x\right] h''(x)\, dx = \int_a^b \alpha(x)\, h''(x)\, dx - C_0 \int_a^b h''(x)\, dx - C_1 \int_a^b x\, h''(x)\, dx$$

$$= 0 - C_0\, [h'(b) - h'(a)] - C_1 \left[x\left(h'(b) - h'(a)\right) - \int_a^b h'(x)\, dx \right]$$

$$= 0 - C_0(0) - C_1\left[(0) - \left(h(b) - h(a)\right)\right] = 0 \qquad (7)$$

On the other hand,

$$\int_a^b \left[\alpha(x) - C_0 - C_1 x\right] h''(x)\, dx = \int_a^b \left[\alpha(x) - C_0 - C_1 x\right]^2 dx \;\; \text{[By (5)]} \qquad (8)$$

By (7) and (8), it follows that

$\alpha(x) - C_0 - C_1 x = 0 \Rightarrow \alpha(x) = C_0 + C_1 x$ for all x in [a, b].

4.2.12. Lemma. If $\alpha(x)$ and $\beta(x)$ are continuous in [a, b] and if

$$\int_a^b \left[\alpha(x)\, h(x) + \beta(x)\, h'(x)\right] dx = 0$$

for every function $h(x) \in D_1[a, b]$ such that h(a) = h(b) = 0 then prove that $\beta(x)$ is differentiable and $\beta'(x) = \alpha(x)$ for all x in [a, b].

Proof. Let us set $A(x) = \int_a^x \alpha(\xi)\, d\xi$. Now integrating by parts, the integral $\int_a^b \alpha(x)\, h(x)\, dx$, we get

$$\int_a^b \alpha(x)\, h(x)\, dx = \left[h(x) \int_a^x \alpha(x)\, dx \right]_a^b - \int_a^b h'(x) \int_a^x \alpha(x)\, dx$$

$$= \left[h(x)\, A(x)\right]_a^b - \int_a^b h'(x)\, A(x)\, dx$$

$$= h(b)\, A(b) - h(a)\, A(a) - \int_a^b h'(x)\, A(x)\, dx = 0 - 0 - \int_a^b h'(x)\, A(x)\, dx$$

$$\Rightarrow \quad \int_a^b \alpha(x)\, h(x)\, dx = -\int_a^b A(x)\, h'(x)\, dx \qquad (1)$$

Now it is given that,

$$\int_a^b [\alpha(x)\, h(x) + \beta(x)\, h'(x)]\, dx = 0$$

Using (1), it becomes

$$\int_a^b [-A(x)\, h'(x) + \beta(x)\, h'(x)]\, dx = 0 \Rightarrow \int_a^b [\beta(x) - A(x)]\, h'(x) = 0 \text{ for all } x \in [a, b]$$

Thus using lemma (2), we get,

$\beta(x) - A(x) = C$, (a constant).

$$\Rightarrow \quad \beta(x) = A(x) + C \Rightarrow \beta(x) = \int_a^x \alpha(\xi)\, d\xi + C \Rightarrow \beta'(x) = \alpha(x) \text{ for all } x \in [a, b]$$

Hence proved.

Remark. The basic work is over. We now introduce the concept of the variation (or differential) of a functional. The concept will the be used to find extrema of functionals.

4.3. Variation of a functional. Let J [y] be a functional defined on some normed linear space and let $\Delta J[h] = J[y + h]$ be its increment corresponding to the increment $h = h(x)$ of the independent variable $y = y(x)$. If y is fixed $\Delta J[h]$ is a functional of h.

Suppose that $\Delta J[h] = \phi[h] + \in \|h\|$ where $\phi[h]$ is a linear functional and $\in \to 0$ as $\|h\| \to 0$. Then the functional J [y] is said to be differentiable and the principal linear part of the increment $\Delta J[h]$ i.e the linear functional $\phi[h]$ is called variation (or differential) of J[y] and is denoted by $\delta J[h]$. Thus, we can write

$$\Delta J[h] = \delta J[h] + \in \|h\| \text{ where } \in \to 0 \text{ as } \|h\| \to 0.$$

4.3.1. Theorem. The differential of a differentiable functional is unique.

Proof. Let $\phi[h]$ be a linear functional such that $\frac{\phi[h]}{\|h\|} \to 0$ as $\|h\| \to 0$, then we claim that $\phi[h] \equiv 0$ for all h.

Let, if possible, $\phi[h_0] \neq 0$ for some $h_0 \neq 0$, then by setting $h_n = \frac{h_0}{n}$ and $\lambda = \frac{\phi[h_0]}{\|h_0\|}$ we observe that

$\|h_n\| \to 0$ as $n \to \infty$ but $\lim_{n\to\infty} \frac{\phi[h_n]}{\|h_n\|} = \lim_{n\to\infty} \frac{n\,\phi[h_0]}{n\,\|h_0\|} = \lambda \neq 0$ which is a contradiction to the hypothesis.

Now, we prove the uniqueness of differential. Let J [y] be a differentiable functional and let, if possible, $\phi_1[h]$ and $\phi_2[h]$ be two variations of J [y], then

$$\Delta J[y] = \phi_1[h] + \in_1 \|h\|; \quad \in_1 \to 0 \text{ as } \|h\| \to 0$$

$$\Delta J[y] = \phi_2[h] + \in_2 \|h\|; \quad \in_2 \to 0 \text{ as } \|h\| \to 0$$

where both $\phi_1[h]$ and $\phi_2[h]$ are linear functionals.

Subtracting these, we get

$$\phi_1[h] - \phi_2[h] = \in_2 \|h\| - \in_1 \|h\| \quad \Rightarrow \frac{\phi_1[h] - \phi_2[h]}{\|h\|} = \in_2 - \in_1$$

Taking limit $\|h\| \to 0$, we get,

$$\lim_{\|h\| \to 0} \frac{\phi_1[h] - \phi_2[h]}{\|h\|} = 0 \qquad [\text{Since } \in_1, \in_2 \to 0 \text{ as } \|h\| \to 0]$$

By above part, we get $\phi_1[h] - \phi_2[h] \equiv 0 \Rightarrow \phi_1[h] = \phi_2[h]$

Hence the uniqueness.

Remark. Let us recall the concept of extremum from analysis.

Let F $(x_1, x_2, \ldots, x_n)$ be a differentiable function of n variables. then $F(x_1, x_2, \ldots, x_n)$

is said to have an extremum at the point $(x_1', x_2', ..., x_n')$ if

$$\Delta F = F(x_1, x_2, \ldots, x_n) - F(x_1', x_2', ..., x_n')$$

has the same sign for all points $(x_1, x_2, \ldots, x_n)$ belonging to some neighbourhood of $(x_1', x_2', ..., x_n')$. Further, the extremum is a minimum if $\Delta F \geq 0$ and is a maximum if $\Delta F \leq 0$.

4.3.2. Extremum of a functional. We say that the functional J [y] has an extremum for y = y if J [y] – J [y] does not change sign in some neighbourhood of the curve y = y (x).

Depending upon whether the functional are the elements of C [a, b] or D[a, b], we define two kinds of extrema:

1. Weak Extremum. We say that the functional J [y] has a weak extremum for y = y if there exists an $\in > 0$ such that J [y] – J [y] has the same sign for all y in the domain of the definition of the functional which satisfy the condition $\|y - y\|_1 < \in$ where $\| \ \|_1$ denotes the norm in D_1 [a, b].

2. Strong Extremum. We say that the functional J [y] has a strong extremum for y = y if there exists an $\in > 0$ such that J [y] – J [y] has the same sign for all y in the domain of definition of the functional which satisfy the condition $\|y - y\|_0 < \in$ where $\| \ \|_0$ denotes the norm in the space C [a, b].

Remark. It is clear by definitions that every strong extremum is simultaneously a weak extremum since if $\|y - y\|_1 < \in$, then $\|y - y\|_0 < \in$ and hence if J [y] is an extremum w.r.t all y such that $\|y - y\|_0 < \in$, then J [y] is certainly an extremum w.r.t. all y such that $\|y - y\|_1 < \in$. However, the converse is not true, in general.

4.3.3. Admissible functions. The set of functions satisfying the constraints of a given variational problem are called admissible functions of that variational problem.

4.3.4. Theorem. A necessary condition for the differentiable functional J [y] to have an extremum for y = y, is that its variation vanish for y = y that is, that δ J [h] = 0 for y = y and all admissible h.

Proof. W.L.O.G., suppose J [y] has a minimum for y = y so that

$$\Delta \ J[h] = J[y + h] - J[y] \geq 0 \text{ for all sufficiently small } \|h\|. \qquad (1)$$

Now by definition we have,

$$\Delta \ J[h] = \delta \ J[h] + \in \|h\| \text{ where } \in \to 0 \text{ as } \|h\| \to 0. \qquad (2)$$

Thus for sufficiently small $\|h\|$, the sign of Δ J [h] will be the same as the sign of δ J [h].

Now, suppose that if possible δ J [h_0] $\neq$ 0 for some admissible h_0.

Then for any $\alpha > 0$, no matter however small,

$$\delta J[-\alpha \ h_0] = -\delta J[\alpha h_0]$$

Thus by (2), ΔJ[h] can be made to have either sign for arbitrary small $\|h\|$ which is a contradiction to (1). Hence δ J [h] = 0 for y = y and all admissible h.

4.3.5. Euler's Equation. Let J [y] be a functional of the form $\int_a^b F(x, y, y')\,dx$ defined on the set of functions y(x) which have continuous first derivatives in [a, b] and satisfy the boundary condition y (a) = A, y(b) = B. Then a necessary condition for J [y] to have an extremum for a given function y(x) is that y(x) satisfy the equation

$$F_y - \frac{d}{dx}(F_{y'}) = 0$$

Proof : Suppose we give y(x) an increment h(x) where in order for the function y(x) + h(x) to continue to satisfy the boundary conditions, we must have h(a) = h(b) = 0.

We calculate corresponding increment to the given functional.

$$\Delta\ J = J\,[y + h] - J\,[y] = \int_a^b F(x\,,\,y + h\,,\,y' + h')\,dx \;-\; \int_a^b F(x\,,\,y\,,\,y')\,dx$$

$$= \int_a^b \left[F\big(x\,,\,y + h\,,\,y' + h'\big) - F\,(x\,,\,y\,,\,y')\right] dx$$

Using Taylor's theorem, we obtain,

$$\Delta\ J = \int_a^b \left[F_y(x\,,\,y\,,\,y')h + F_{y'}(x\,,\,y\,,\,y')h'\right] dx \;+ \begin{bmatrix}\text{terms containing higher} \\ \text{order partial derivatives} \\ \text{and powers of h and } h' \\ \text{greater than 1}\end{bmatrix}$$

We express this as

$$\Delta\ J = \int_a^b \left[F_y(x\,,\,y\,,\,y')h + F_{y'}(x\,,\,y\,,\,y')h'\right] dx \;+ \ldots$$

Clearly the integral $\int_a^b \left[F_y(x\,,\,y\,,\,y')h + F_{y'}(x\,,\,y\,,\,y')h'\right] dx$ represents the principal linear part of Δ J and hence, we write,

$$\delta\ J = \int_a^b \left[F_y(x\,,\,y\,,\,y')h + F_{y'}(x\,,\,y\,,\,y')h'\right] dx$$

Now by theorem (2), the necessary condition for J [y] to be extremum is that δ J = 0, so that

$$\int_a^b \left[F_y(x\,,\,y\,,\,y')h + F_{y'}(x\,,\,y\,,\,y')h'\right] dx = 0$$

By lemma (4) (proved earlier), we obtain that

$$F_y = \frac{d}{dx}\,(F_{y'}) \qquad [\text{Take } F_y = \alpha\,(x) \text{ and } F_{y'} = \beta\,(x) \text{ in lemma (4)}]$$

$$\Rightarrow \qquad F_y - \frac{d}{dx}\,(F_{y'}) = 0.$$

This equation is known as Euler's Equation.

4.3.6. Another form of Euler's equation : As F is a function of x, y and y', so we have :

$$\frac{dF}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\frac{dy'}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y'' \quad (1)$$

Also we have,

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'}y'' \quad (2)$$

Subtracting (2) from (1), we obtain,

$$\frac{dF}{dx} - \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y}\right)$$

which can be written as,

$$\frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] - \frac{\partial F}{\partial x} = y'\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] = y'\,[0] \text{ [By Euler's equation]}$$

$$\Rightarrow \quad \frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] - \frac{\partial F}{\partial x} = 0 \text{ or } \frac{d}{dx}\left(F - y'F_{y'}\right) - F_x = 0$$

This is another form of Euler equation.

Remark. Euler's equation

$$F_y - \frac{d}{dx}(F_{y'}) = 0 \quad (*)$$

and

$$\frac{d}{dx}\left(F - y'F_{y'}\right) - F_x = 0 \quad (**)$$

plays a fundamental role in the calculus of variations and is in general a second order differential equation we now discuss some special cases:

Case 1. Suppose the integrand does not depend on y that is, let the functional of under consideration has the form $\int_a^b F(x, y')\,dx$, so that we have $F_y = 0$

Then by (*), $\frac{d}{dx}(F_{y'}) = 0 \quad \Rightarrow \; F_{y'} = C$, a constant which is a first order differential equation and can be solved by integration.

Case 2. If the integrand does not depend on x that is, the functional has the form

$\int_a^b F(y, y')\,dx$ then $F_x = 0$ and so by (**), $\frac{d}{dx}\left(F - y'F_{y'}\right) = 0 \quad \Rightarrow \; F - y'F_{y'} = C$

Case 3. If F does not depend upon y', then again by (*), we get, $F_y = 0$ which is not a differential equation but a finite equation whose solution consists of one or more curves $y = y(x)$.

Case 4. Consider the functional of the form

$$\int_a^b f(x, y)\, ds = \int_a^b f(x, y)\sqrt{1+y'^2}\, dx$$

representing the integral of a function f (x, y) w.r.t. the arc length s where $ds = \sqrt{1+y'^2}\,dx$. In this case we have $F(x, y, y') = f(x, y)\sqrt{1+y'^2}$ and Euler's equation becomes,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = f_y(x, y)\sqrt{1+y'^2} - \frac{d}{dx}\left[f(x, y)\frac{y'}{\sqrt{1+y'^2}}\right]$$

$$= f_y\sqrt{1+y'^2} - f_x\frac{y'}{\sqrt{1+y'^2}} - f_y\frac{y'^2}{\sqrt{1+y'^2}} - f\frac{y''}{\left(1+y'^2\right)^{3/2}} = \frac{f_y}{\sqrt{1+y'^2}} - \frac{f_x y'}{\sqrt{1+y'^2}} - \frac{f\,y''}{\left(1+y'^2\right)^{3/2}}$$

$= 0$

Thus, $f_y - f_x\, y' - f\,\dfrac{y''}{1+y'^2} = 0$ which is the required form of Euler's equation.

4.3.7. Example. Find the extremal of the functional $J[y] = \int_1^2 \frac{\sqrt{1+y'^2}}{x}\,dx$ $y(1) = 0$, $y(2) = 1$.

Solution. Since the integrand does not contain y, so we shall use Euler's equation in the form

$$F_{y'} = \text{constant} = c \text{ (say)} \qquad (*)$$

Now, we have

$$F = \frac{\sqrt{1+y'^2}}{x} \Rightarrow F_{y'} = \frac{1}{2}\cdot\frac{2y'}{\sqrt{1+y'^2}}\cdot\frac{1}{x} = \frac{y'}{x\sqrt{1+y'^2}}$$

Using this in (*), we get

$$\frac{y'}{x\sqrt{1+y'^2}} = c \Rightarrow \left(\frac{dy}{dx}\right)^2 = c^2x^2\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

$$\Rightarrow (1 - C^2x^2)\left(\frac{dy}{dx}\right)^2 = c^2x^2 \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{c^2x^2}{1-c^2x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{cx}{\sqrt{1-c^2x^2}}$$

$$\Rightarrow \quad y = \int \frac{cx}{\sqrt{1-c^2x^2}} + c' = \int \frac{x}{\sqrt{\frac{1}{c^2}-x^2}}\,dx + c' = -\frac{1}{2}\int \frac{-2x}{\sqrt{\frac{1}{c^2}-x^2}}\,dx + c'$$

$$= -\frac{1}{2}\int \frac{\left(\frac{1}{c^2}-x^2\right)^{1/2}}{1/2} + c' = -\sqrt{\frac{1}{c^2}-x^2} + c'$$

$$\Rightarrow \quad (y - c')^2 = \frac{1}{c^2} - x^2 \Rightarrow (y - c')^2 + x^2 = k^2, \text{ say}$$

Thus, the solution is a circle with its centre on the y – axis. Using the conditions

$$y(1) = 0,\ y(2) = 1,$$

we find that $c' = 2$, $k = \sqrt{5}$

So that the final solution is, $(y-2)^2 + x^2 = 5$.

4.3.8. Example. Among all the curves joining two given points (x_0, y_0) and (x_1, y_1). Find the one which generates the surface of minimum area when rotaed about the x – axis.

Solution. We know (from calculus) that the area of surface of revolution generated by rotating the curve $y = y(x)$ about the x – axis is given by :

$$S = \int_{x_0}^{x_1} 2\pi\, y\sqrt{1+y'^2}\,dx = 2\pi \int_{x_0}^{x_1} y\sqrt{1+y'^2}\,dx$$

Since the integrand does not depend explicitly on x, Euler's equation can be written as :

$$F - y' F_{y'} = C, \text{ constant.}$$

$$\Rightarrow \quad y\sqrt{1+y'^2} - y\frac{y'^2}{\sqrt{1+y'^2}} = C \quad \Rightarrow y\left[\frac{1+y'^2-y'^2}{\sqrt{1+y'^2}}\right] = C \Rightarrow \frac{y^2}{1+y'^2} = C^2$$

$$\Rightarrow \quad y' = \frac{\sqrt{y^2-C^2}}{C} \Rightarrow \frac{dy}{dx} = \frac{1}{C}\sqrt{y^2-C^2} \Rightarrow dx = \frac{C\,dy}{\sqrt{y^2-C^2}}$$

$$\Rightarrow \quad x + C_1 = C\cosh^{-1}\left(\frac{y}{C}\right) \Rightarrow y = C\cosh\left(\frac{x+C_1}{C}\right)$$

which is the equation of a catenary. The values of arbitrary constants can be determined by the conclitions

$$y(x_0) = y_0,\ y(x_1) = y_1$$

4.3.9. Example. Find the extremal of the functional $J[y] = \int_a^b (x-y)^2 dx$

Solution. Here $F = (x - y)^2$ which does not contain y' explicitely so that Euler's equation is $F_y = 0$ which gives

$$2(x-y) = 0 \qquad \Rightarrow y = x$$

which is a finite equation and represents a straight line.

4.3.10. Example. Show that the shortest distance between two points in a plane is a straight line.

Solution. Let A (x_1, y_1) and B (x_2, y_2) be the given points and let 's' be the length of curve connecting them, then $S = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$

Here $F = \sqrt{1+y'^2}$ which is independent of y. So Euler's equation is

$$F_{y'} = \text{constant} \Rightarrow \frac{y'}{\sqrt{1+y'^2}} = c \Rightarrow \frac{y'^2}{c^2} = 1+y'^2 \Rightarrow y'^2 = \frac{c^2}{1-c^2} \Rightarrow y' = \frac{c}{\sqrt{1-c^2}} = m \text{ (say)}$$

$$\Rightarrow \quad \frac{dy}{dx} = m \qquad \Rightarrow \qquad y = mx + c,$$ which is the equation of a straight line.

4.3.11. Exercise.

1. Show that the functional $\int_0^1 (xy + y^2 - 2y^2 y')\, dx$, $y(0) = 1$, $y(1) = 2$ cannot have any stationary function.
2. Find extremals of the functional $J[y(x)] = \int_0^{2\pi} (y'^2 - y^2)\, dx$ that satisfy the boundary condition $y(0) = 1$, $y(2\pi) = 1$.

Answer. $y = 1. \cos x + C_2 \sin x$ that is, $y = \cos x + C_2 \sin x$.

3. Obtain the general solution of the Euler's equation for the functional $\int_a^b \frac{1}{y} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

Answer. $(x-h)^2 + y^2 = k^2$.

4. Find the curve on which functional $\int_0^1 \left[(y')^2 + 12xy\right] dx$ with boundary conditions $y(0) = 0$ and $y(1) = 1$ can be extremized.

Answer. $y = x^3$.

4.4. Functionals Dependent on Higher Order Derivatives.

4.4.1. Theorem. A necessary condition for the extremum of a functional of the form $J[y]=\int_a^b F\left[x, y, y', ..., y^{(n)}\right] dx$, where F is differentiable w.r.t. each of its arguments is

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} + \ldots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0$$

Proof. Let us consider the functional,

$$J[y] = \int_a^b F\left[x, y, y', ..., y^{(n)}\right] dx$$ satisfying the boundary conditions,

$$y(a) = y_1;\ y'(a) = y_1';\ \ldots;\ y^{(n-1)}(a) = y_1^{n-1}$$

$$y(b) = y_2;\ y'(b) = y_2';\ \ldots;\ y^{(n-1)}(b) = y_2^{n-1}$$

We give y(x) an increment h(x) so that y(x) + h(x) also satisfies the above boundary conditions. For this, we must have

$$h(a) = h'(a) = \ldots = h^{(n-1)}(a) = 0$$

and $$h(b) = h'(b) = \ldots = h^{(n-1)}(b) = 0 \qquad (1)$$

We now calculate the corresponding increment to the given functional,

$$\Delta J = J[y+h] - J[y]$$

which gives,

$$\Delta J = \int_a^b \left[F\left(x, y+h, y'+h', ..., y^{(n)}+h^{(n)}\right) - F\left(x, y, y', ..., y^{(n)}\right)\right] dx$$

Using Taylor's theorem, we obtain,

$$\Delta J = \int_a^b \left(F_y h + F_{y'} h' + ... + F_{y^{(n)}} h^{(n)}\right) dx + ...$$

The integral on R.H.S. represents the principal linear part of the increment ΔJ and hence the variation of J [y] is

$$\delta J = \int_a^b \left(F_y h + F_{y'} h' + ... + F_{y^{(n)}} h^{(n)}\right) dx$$

Therefore, the necessary condition $\delta J = 0$ for an extemum implies that

$$\int_a^b \left(F_y h + F_{y'} h' + ... + F_{y^{(n)}} h^{(n)}\right) dx = 0 \qquad (2)$$

On R.H.S. of (2), integrate the 2nd term by parts once so that,

$$\int_a^b F_{y'}\, h'\, dx = \left[F_{y'}\, h(x)\right]_a^b - \int_a^b \frac{d}{dx}\left(F_{y'}\right) h(x)\, dx \qquad (3)$$

On R.H.S. of (2), integrated the 3rd term by parts twice to get

$$\int_a^b F_{y''} h'' dx = \left[F_{y''}\, h'(x)\right]_a^b - \left[\frac{d}{dx}\left(F_{y''}\right) h(x)\right]_a^b + \int_a^b \frac{d^2}{dx^2}\left(F_{y''}\right) h\, dx \quad (4)$$

Continuing like this, integrating the last term by parts n times, we get

$$\int_a^b F_{y^{(n)}} h^{(n)}\, dx = \left[F_{y^{(n)}}\, h^{(n-1)}(x)\right]_a^b - \left[\frac{d}{dx}\left(F_{y^{(n)}}\right) h^{(n-2)}(x)\right]_a^b + \ldots + (-1)^n \int_a^b \frac{d^n}{dx^n}\left(F_{y^{(n)}}\right) h(x)\, dx \qquad (5)$$

Using the boundary conditions (1) in (3), (4), (5), the integrated parts within the limits a and b vanish and then using these in (2), we get

$$\int_a^b \left[F_y - \frac{d}{dx}\left(F_{y'}\right) + \frac{d^2}{dx^2}\left(F_{y''}\right) + \ldots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}}\right] h(x)\, dx = 0$$

Thus by a lemma proved earlier, we have

$$F_y - \frac{d}{dx}\left(F_{y'}\right) + \frac{d^2}{dx^2}\left(F_{y''}\right) + \ldots + (-1)^n \frac{d^n}{dx^n}\left(F_y(n)\right) = 0$$

This result is again called Euler's equation which is a differential equation of order 2n. Its general solution contains 2n arbitrary constants which can be determined by the boundary conditions.

4.4.2. Example. Find the stationary function of the functional $\int_a^b (y'^2 + y\,y'')\, dx$; $y(a) = \lambda_1$ $y'(a) = \lambda_2$, $y(b) = \lambda_3$ $y'(b) = \lambda_4$.

Solution. The given functional is $\int_a^b (y'^2 + y\,y'')\, dx$. Let $F(x, y, y', y'') = y'^2 + y\,y''$.

The Euler's - Poisson Equation is $F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} = 0$ (1)

Here $F_y = y''$ $F_{y'} = 2y'$ $F_{y''} = y$.

So (1) $\Rightarrow$ $y'' - \frac{d}{dx}(2y') + \frac{d^2}{dx^2}(y) = 0$

$y'' - 2y'' + y'' = 0$ $\Rightarrow$ 0 = 0, which is not a differential equation.

So, there is no extremal and hence no stationary function.

4.4.3. Exercise.

1. Find the curve the extremises the functional $\int_0^{\pi/4}(y''^2 - y^2 + x^2)\,dx$ under the boundary condition y(0) = 0, $y'(0) = 1$, $y(\pi/4) = y'(\pi/4) = \frac{1}{\sqrt{2}}$.

Answer. y = sinx

4.5. Functionals dependent on Functions of Several Independent Variables.

So far, we have considered functionals depending on functions of one variable that is, on curves. In many problems, one encounters functionals depending on functions of several independent variables that is, on surfaces. We now, try to find the extremum of such functionals.

However, for simplicity, we confine ourselves to the case of two independent variables. Thus, let F(x, y, z, z_x, z_y) be a function with continuous first and second partial derivatives w.r.t all its arguments and consider a functional of the form

$$J(z) = \iint_R F(x, y, z, z_x, z_y)\,dx\,dy$$

where R is some closed region. Before giving the Euler's equation for such functionals, we prove the following lemma.

4.5.1. Lemma. If α (x, y) is a fixed function which is continuous in a closed region R and if the integral

$$\iint_R \alpha(x, y)\, h(x, y)\,dx\,dy$$

vanishes for every function h (x, y) which has continuous first and second derivatives in R and equals zero on the boundary D of R, then prove that α (x, y) = 0 everywhere in R.

Proof. Let, if possible, the function α (x, y) is non zero, say positive at some point say (x_0, y_0) in R. Then by continuity α (x, y) is also positive in some circle

$$(x - x_0)^2 + (y - y_0)^2 \le \in^2 \qquad (1)$$

contained in R with centre (x_0, y_0) and radius $\in$.

Now we set h(x, y) = $\left[(x - x_0)^2 + (y - y_0)^2 - \in^2\right]^3$ inside the circle (1) and h (x, y) = 0 outside the circle. It is clear that h (x, y) has continuous first and second order derivatives in circle and also h(x) equals zero on the boundary of the circle so that all the conditions of the lemma are satisfied. Hence we must have

$$\iint_{R'} \alpha(x, y)\, h(x, y)\,dx\,dy = 0 \qquad (2)$$

where R' is circle given by (1)

But it is clear that integrand in (2) is positive in circle (1) and so integral (2) is obviously positive.

This contradiction proves the lemma.

Green's Theorem. It states that

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int_D Pdx + Qdy ,$$

where D is boundary of the surface R.

4.5.2. Theorem. A necessary condition for the functional

$$J(z) = \iint_R F(x, y, z, z_x, z_y) dx\, dy$$

to have an extremum for a given function z (x, y) is that z (x, y) satisfies the equation

$$F_z - \frac{\partial}{\partial x} F_{zx} - \frac{\partial}{\partial y} F_{zy} = 0$$

Proof. Let h (x, y) be an arbitrary function which has continuous first and second derivatives in the region R and vanishes on the boundary D of R. Then if z (x, y) belongs to the domain of definition of the given functional so does z (x, y) + h (x, y)

Thus we have,

$$\Delta J = J[z + h] - J[z] = \iint_R F(x, y, z + h, z_x + h_x, z_y + h_y) - F(x, y, z, z_x, z_y)\, dxdy$$

Using Taylor's theorem, we get,

$$\Delta J = \iint_R (F_z h + F_{z_x} h_x + F_{z_y} h_y)\, dxdy + \ldots$$

Integral on R.H.S. represents the principal linear part of the increment ΔJ and hence the variation of J[z] is,

$$\delta J = \iint_R (F_z h + F_{z_x} h_x + F_{z_y} h_y)\, dxdy \qquad (1)$$

Now consider,

$$\frac{\partial}{\partial x}\left(F_{z_x} h\right) = F_{z_x} h_x + \frac{\partial}{\partial x}\left(F_{z_x}\right).h \text{ and } \frac{\partial}{\partial y}\left(F_{z_y} h\right) = F_{z_y} h_y + \frac{\partial}{\partial y}\left(F_{z_y}\right).h$$

which give, $F_{z_x} h_x = \frac{\partial}{\partial x}\left(F_{z_x} h\right) - \frac{\partial}{\partial x}\left(F_{z_x}\right).h$ and $F_{z_y} h_y = \frac{\partial}{\partial y}\left(F_{z_y} h\right) - \frac{\partial}{\partial y}\left(F_{z_y}\right).h$

Using these in (1), we obtain

$$\delta J = \iint_R F_z h\, dxdy + \iint_R \left[\frac{\partial}{\partial x}\left(F_{z_x} h\right) + \frac{\partial}{\partial y}\left(F_{z_y} h\right) \right] dxdy - \iint_R \left[\frac{\partial}{\partial x}\left(F_{z_x} 1 + \frac{\partial}{\partial y} F_{z_y} h \right) \right] h\, dxdy$$

$$= \iint_R F_z h\, dxdy + \int_D \left(F_{z_x} h\, dy - F_{z_y} h\, dx\right) - \iint_R \left[\frac{\partial}{\partial x}\left(F_{z_x}\right) + \frac{\partial}{\partial y}\left(F_{z_y}\right)\right] h\, dxdy\,,$$

using Green's theorem

The second integral on R.H.S. is zero because h(x, y) vanishes on the boundary D and hence, we obtain,

$$\delta\, J = \iint_R \left[F_z - \frac{\partial}{\partial x}\left(F_{z_x}\right) - \frac{\partial}{\partial y}\left(F_{z_y}\right)\right] h(x\,,y)\, dx\, dy$$

Thus, the condition for extremum, $\delta\, J = 0$ implies that

$$\iint_R \left[F_z - \frac{\partial}{\partial x}\left(F_{z_x}\right) - \frac{\partial}{\partial y}\left(F_{z_y}\right)\right] h(x\,,y)\, dx\, dy = 0$$

Hence by lemma (5), we have,

$$F_z - \frac{\partial}{\partial x}\left(F_{z_x}\right) - \frac{\partial}{\partial y}\left(F_{z_y}\right) = 0$$

which is the required condition. This equation is known as Euler's equation and is a second order partial differential equation in general.

4.5.3. Example. Derive Euler's equation for the functional

$$J\,[z] = \iint_R \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 dx\, dy$$

Solution. Here, $F\,(x, y, z, z_x, z_y) = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(z_x\right)^2 - \left(z_y\right)^2$. Therefore

$$F_z = 0,\; F_{z_x} = 2z_x,\; F_{z_y} = -2z_y$$

Now, the Euler's equation is,

$$F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y} = 0 \quad \Rightarrow \quad 0 - \frac{\partial}{\partial x}(2\, z_x) + \frac{\partial}{\partial y}(2\, z_y) = 0$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial y^2} - \frac{\partial^2 z}{\partial x^2} = 0$$

which is the required Euler's equation. The solution of this second order partial differential equation will give the extremal of the given functional.

4.5.4. Example. Find the surface of least area spanned by a given contour.

Solution. In this case, we have to find the minimum of the functional.

$$J\,[z] = \iint_R \sqrt{1 + z_x^2 + z_y^2}\; dxdy$$

$$\Rightarrow \qquad F(x, y, z, z_x, z_y) = \sqrt{1 + z_x^2 + z_y^2}$$

Therefore, $F_z = 0$; $F_{z_x} = \dfrac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}$; $F_{z_y} = \dfrac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}$

Now, the Euler's equation is

$$F_z - \frac{\partial}{\partial x}\left(F_{z_x}\right) - \frac{\partial}{\partial y}\left(F_{z_y}\right) = 0$$

$$\Rightarrow \qquad \frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0 \qquad (1)$$

Let us calculate,

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) = \frac{\left(1+p^2+q^2\right)r - p^2r - pqs}{\left(1+p^2+q^2\right)^{3/2}}$$

where $z_x = p$; $z_y = q$; $z_{xx} = r$; $z_{xy} = z_{yx} = s$; $z_{yy} = t$

Similarly

$$\frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = \frac{\left(1+p^2+q^2\right)t - q^2t - pqs}{\left(1+p^2+q^2\right)^{3/2}}$$

Using these in (1), we have,

$$\left(1+p^2+q^2\right) r - p^2r - pqs + \left(1+p^2+q^2\right) t - q^2t - pqs = 0$$

$$\Rightarrow \qquad r(1+q^2) - 2pqs + t(1+p^2) = 0$$

The solution of this differential equation will provide the solution.

4.5.5. Example. Show that the functional $\int_0^1 \left[2x + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] dt$ with x(0) = 1 y(0) = 1, x(1) = 1.5 y(1) = 1 is stationary for $x = \dfrac{2+t^2}{2}$, y = 1.

Solution. The given functional is $\int_0^1 (2x + y'^2 + x'^2)\, dx$

Let $F = 2x + x'^2 + y'^2$

The Euler's equations are $\qquad F_x - \dfrac{d}{dt} F_{x'} = 0 \qquad (1)$

$$F_y - \frac{d}{dt} F_{y'} = 0 \qquad (2)$$

Here $F_x = 2$ $F_y = 0$ $F_{x'} = 2x'$ $F_{y'} = 2y'$

So, (1) $\Rightarrow 2 - \frac{d}{dt}(2x') = 0 \Rightarrow \frac{d^2x}{dt^2} = 1 \Rightarrow \frac{dx}{dt} = t + C_1$

$$\Rightarrow \qquad x = \frac{t^2}{2} + C_1 t + C_2 \qquad (3)$$

Also (2) $\Rightarrow 0 - \frac{d}{dt}(2y') = 0 \Rightarrow \frac{dy'}{dt} = 0 \Rightarrow y' = C_3$

$$\Rightarrow \qquad y = C_3 t + C_4 \qquad (4)$$

So, (3) and (4) are equations of extremals

The boundary conditions are x(0) = 1 y(0) = 1 x(1) = 1.5 y(1) = 1

$x(0) = 1 \Rightarrow 0 + C_1(0) + C_2 = 1 \Rightarrow C_2 = 1$

$y(0) = 1 \Rightarrow C_3(0) + C_4 = 1 \Rightarrow c_4 = 1$

$x(1) = 1.5 \Rightarrow \frac{1}{2} + C_1(1) + C_2 = 1.5 \Rightarrow \frac{1}{2} + C_1 + 1 = 1.5 \Rightarrow C_3 = 0$

So, $C_1 = 0$, $C_2 = 1$, $C_3 = 0$, $C_4 = 1$

So, equation (3) $\Rightarrow x = \frac{t^2}{2} + 0.t + 1$, that is, $x = \frac{2+t^2}{2}$

Equation (4) $\Rightarrow y = 0.t + 1$, that is, y = 1

Therefore, stationary functions are $x = \frac{2+t^2}{2}$, y = 1.

4.6. Variable End Point Problem. So far, we have discussed the functionals with fixed end points. Sometimes it may happen that the end points lie on two given curves

$$y = \phi(x) \text{ and } y = \Psi(x).$$

Such problems are called variable end point problems. We discuss only a particular case in the form of following problem.

4.6.1. Problem. Among all curves whose end points lie on two given vertical lines x = a and x = b. Find the curve for which the functional

$$J[y] = \int_a^b F(x, y, y')\, dx \qquad (1)$$

has an extremum.

Solution. As before, we calculate

$$\Delta J = J[y+h] - J[y] = \int_a^b [F(x, y+h, y'+h') - (x, y, y')]dx$$

Using Taylor's theorem, we obtain

$$\Delta J = \int_a^b (F_y h + F_{y'} h')\, dx + \ldots$$

Then the variation δJ of the functional J[y] is given by principal linear part of ΔJ that is,

$$\delta J = \int_a^b (F_y h + F_{y'} h')\, dx$$

Here, unlike the fixed end point problem, h(x) need no longer vanish at the points a and b, so that integrating by parts the second term, we get

$$\delta J = \int_a^b \left(F_y - \frac{d}{dx} F_{y'} \right) h(x)\, dx + \left[F_{y'} h(x) \right]_{x=a}^{x=b}$$

$$= \int_a^b \left(F_y - \frac{d}{dx} F_{y'} \right) h(x)\, d(x) + F_{y'}\Big|_{x=b} h(b) - F_{y'}\Big|_{x=a} h(a) \qquad (2)$$

We first consider functions h(x) such that h(a) = h(b) = 0. Then, as in simplest variational problem, the condition $\delta J = 0$ implies that

$$F_y - \frac{d}{dx} F_{y'} = 0 \qquad (3)$$

Thus, in order for the curve y = y(x) to be a solution of the variable end point problem, y must be an extremal that is, a solution of the Euler's equation. But if y is an extremal, the integral in the expression (2) for δ J vanishes and then the condition δ J = 0 takes the form,

$$F_{y'}\Big|_{x=b} h(b) - F_{y'}\Big|_{x=a} h(a) = 0$$

But since h(x) is arbitrary, it follows that

$$F_{y'}\Big|_{x=a} = 0 \text{ and } F_{y'}\Big|_{x=b} = 0 \qquad (4)$$

Thus, to solve the variable end point problem we must first find a general integral of Euler's equation (3) and then use the condition (4) to determine the values of arbitrary constants.

Remark.

1. The conditions (4) are some times called the natural boundary conditions.

2. Besides the case of fixed end points and the case of variable end points, we can also consider the mixed case, where one end is fixed and the other is variable.

For example, suppose we are looking for an extemum of the functional (1) w.r.t. the class of curves joining a given point A (with abscissa a) and an arbitrary point of the line x = b. In this case, the conditions (4) reduce to the single condition $F_{y'}\big|_{x=b} = 0$ and y (a) = A serves as the second boundary condition.

4.6.2. Example. Starting from the point P (a, A), a heavy particle slides down a curve in a vertical plane. Find the curve such that the particle reaches the vertical line x = b ($\neq$ a) in the shortest time.

Solution. For simplicity, we assume the point P to be origin. Then velocity of the particle,

$$v = \frac{ds}{dt} = \sqrt{1+y'^2}\,\frac{dx}{dt} \Rightarrow dt = \frac{\sqrt{1+y'^2}}{v}\,dx$$

Also, we have, $v^2 - u^2 = 2gh \Rightarrow v^2 - 0 = 2gy \Rightarrow v = \sqrt{2gy}$

So that $dt = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$ or the total time, $T = \int \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$

We have to find least value of T. In this case, $F = \sqrt{\frac{1+y'^2}{2gy}}$

Since x is absent, so Euler's equation is taken as $F - y' F_{y'}$ = constant = C (say)

$$\Rightarrow \quad \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} - y' \cdot \frac{y'}{\sqrt{2gy}\sqrt{1+y'^2}} = C \Rightarrow \frac{1+y'^2-y'^2}{\sqrt{1+y'^2}\sqrt{2gy}} = C \Rightarrow C^2\left(1+y'^2\right) = \frac{1}{2gy}$$

$$\Rightarrow \quad y'^2 = \frac{K}{y} - 1 \text{ where } \frac{1}{2gc^2} = K \Rightarrow \frac{dy}{dx} = \sqrt{\frac{K-y}{y}} \text{ or } \sqrt{\frac{y}{K-y}}\,dy = dx$$

Integrating, we get, $\int \frac{\sqrt{y}}{\sqrt{K-y}}\,dy = \int dx + C_1$ (1)

Let $y = K\sin^2 \theta/2 \Rightarrow dy = K \sin\theta/2\,\cos\theta/2\,d\theta$

so that (1) becomes

$$\int \frac{\sqrt{K}\sin\theta/2}{\sqrt{K}\cos\theta/2}.\,K \sin\theta/2\,\cos\theta/2\,d\theta = x + C_1$$

$$\int K\sin^2\theta/2\,d\theta = x + C_1 \Rightarrow K\int \frac{1-\cos\theta}{2}\,d\theta\; x + C_1 \Rightarrow \frac{K}{2}(\theta - \sin\theta) = x + C_1$$

Or $\quad x = \frac{K}{2}(\theta - \sin\theta) - C_1$

Again since y = K $\sin^2\theta/2 \Rightarrow y = \frac{K}{2}(1-\cos\theta)$

Thus the solution is, $x = \frac{K}{2}(\theta-\sin\theta) - C_1$; $y = \frac{K}{2}(1-\cos\theta)$

Since the curve pass through the origin so $C_1 = 0$

So that the curve is $x = r(\theta-\sin\theta)$, $y = r(1-\cos\theta)$, $r = \frac{K}{2}$

This is an equation of a cycloid and value of r is determined by the second condition, namely

$$F_{y'}\Big|_{x=b} = 0 \quad \Rightarrow \quad \frac{y'}{\sqrt{2gy}\sqrt{1+y'^2}} = 0 \text{ for } x = b$$

$$\Rightarrow y' = 0 \text{ for } x = b \Rightarrow \frac{dy}{dx} = 0 \text{ for } x = b$$

Now $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{1-\cos\theta} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = \pi$ for x = b

Now $x = r(\theta-\sin\theta) \Rightarrow r = \frac{x}{\theta-\sin\theta} = \frac{b}{\pi-\sin\pi} = \frac{b}{\pi}$

Hence the required curve finally comes as

$$x = \frac{b}{\pi}(\theta-\sin\theta)\ ;\ y = \frac{b}{\pi}(1-\cos\theta)$$

4.7. Variational Derivatives. We introduce the variational (or functional) derivative, which plays the same role for functionals as the concept of the partial derivative plays for functions of n variables. We shall follow the approach to first go from the variational problem to an n-dimensional problem and then pass to the limit $n \to \infty$.

Remark. From elementary analysis we know that a necessary condition for a function of n variables to have an extremum is that all its partial derivatives vanish. Now we derive the corresponding condition for funcionals.

Consider the functional,

$$J[y] = \int_a^b F(x, y, y')\,dx \qquad (1)$$

$$y(a) = A,\ y(b) = B$$

Divide the interval [a, b into n + 1 equal subintervals by introducing the points

$$A = x_0, x_1, x_2, \ldots, x_n, x_{n+1} = b$$

and we replace the smooth function y(x) by the polygonal line with vertices

$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n), (x_{n+1}, y_{n+1})$ or $(a, A), (x_1, y_1), \ldots, (x_n, y_n), (b, B)$

where $y_i = y(x_i)$. We shall denote $x_{i+1} - x_i$ by Δx. Then we approximate (1) by the sum

$$J[y_1, y_2, \ldots, y_n] = \sum_{i=0}^{n} F\left(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}\right)\Delta x \tag{2}$$

which is a function of n variables.

Now, we calculate the partial derivatives $\dfrac{\partial J(y_1, y_2, \ldots, y_n)}{\partial y_k}$

We observe that each variable y_k in (2) appears in just two terms corresponding to $i = k - 1$ and $i = k$ and these two terms are,

$$F\left(x_{k-1}, y_{k-1}, \frac{y_k - y_{k-1}}{\Delta x}\right)\Delta x + F\left(x_k, y_k, \frac{y_{k+1} - y_k}{\Delta x}\right)\Delta x$$

Thus we have,

$$\frac{\partial J}{\partial y_k} = F_y\left(x_k, y_k, \frac{y_{k+1} - y_k}{\Delta x}\right)\Delta x + F_{y'}\left(x_{k-1}, y_{k-1}, \frac{y_k - y_{k-1}}{\Delta x}\right) - F_{y'}\left(x_k, y_k, \frac{y_{k+1} - y_k}{\Delta x}\right)\Delta x \tag{3}$$

Dividing (3) by Δx, we get

$$\frac{\partial J}{\partial y_k \Delta x} = F_y\left(x_k, y_k, \frac{y_{k+1} - y_k}{\Delta x}\right) - \frac{1}{\Delta x}\left[F_{y'}\left(x_k, y_k, \frac{y_{k+1} - y_k}{\Delta x}\right) - F_{y'}\left(x_{k-1}, y_{k-1}, \frac{y_k - y_{k-1}}{\Delta x}\right)\right] \tag{4}$$

The expression $\partial y_k \, \Delta x$ appearing in denominator on the left has a direct geometric meaning.

As $\Delta x \to 0$, the expression (4) gives

$$\frac{\delta J}{\delta y} \equiv F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') \qquad \ldots\ldots\ldots(*)$$

This is known as the variational derivative of the functional.

Remark. By equation (*), we observe that variational derivative $\dfrac{\delta J}{\delta y}$ is just the left – hand side of Euler's equation and for extremal variational derivative of the functional under consideration should vanish at every point. This is the analog of the situation encountered in elementary analysis, where a necessary condition for a function of n variables to have an extremum is that all in partial derivatives vanish.

4.7.1. Another definition of variational derivative. Let J [y] be a functional depending on the function y(x) and suppose we give y(x) and increment h(x) which is non zero only in the neighbourhood of a point x_0. Let Δ_σ be the area lying between the curve y = y(x) and y = y(x) + h(x).

Now, dividing the increment J(y + h) − J [y] of the functional J [y] by the area Δ_σ we obtain the ratio,

$$\frac{J[y+h] - J[y]}{\Delta_\sigma} \qquad (1)$$

Now, we let the area Δ_σ tend to zero in such a way that both max. $|h(x)|$ and the length of the interval in which h(x) is non –zero tend to zero. Then, if the ratio (1) converges to a limit as $\Delta\sigma \to 0$, this limit is called the variational derivative of the functional J[y] at the point x_0 and is denoted by $\left.\frac{\delta J}{\delta y}\right|_{x=x_0}$.

Remark.

1. It is clear from the definition of the variational derivative that the increment

$$\Delta J \equiv J[y+h] - J[y] = \left\{\left.\frac{\delta J}{\delta y}\right|_{x=x_0} + \in\right\}\Delta\sigma$$

where $\in \to 0$ as both max. $|h(x)|$ and the length of the interval in which h(x) is non vanishing tend to zero.

2. It also follows that in terms of variational derivative, the differential or variation of the functional J[y] at the point x_0 is given by

$$\delta J = \left.\frac{\delta J}{\delta y}\right|_{x=x_0} \Delta\sigma$$

4.7.2. Invariance of Euler's Equation. In this section we show that whether or not a curve is an extremal is a property which is independent of the choice of the coordinate system.

For this, consider the functional $J[y] = \int_a^b F(x, y, y')\,dx$ (1)

Now we introduce another system of coordinates by substituting,

x = x (u, v) and y = y (u, v) such that the Jacobian $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0$ (2)

Then the curve given by the equation y = y(x) in the xy, plane corresponds to the curve given by some equation

v = v(u) in the uv – plane.

Now, we have

$$\frac{dx}{du} = \frac{\partial x}{\partial u}\frac{du}{du} + \frac{\partial x}{\partial v}.\frac{dv}{du} = x_u + x_v$$

$$\frac{dy}{du} = \frac{\partial y}{\partial u}\frac{du}{du} + \frac{\partial y}{\partial v}.\frac{dv}{du} = y_u + y_v\, v'$$

which gives

$$\frac{dy}{dx} = \frac{y_u + y_v\, v'}{x_u + x_v\, v'} \text{ ad } dx = (x_u + x_v\ v')du$$

By these substitutions, functional (1) transforms into the functional,

$$J_1[v] = \int_{a_1}^{b_1} F\left[x(u, v), y(u, v), \frac{y_u + y_v\, v'}{x_u + x_v\, v'}\right](x_u + x_v\ v')du = \int_{a_1}^{b_1} F_1\ (u, v, v')du\ \text{(say)} \qquad (3)$$

We now show that if y = y(x) satisfies the Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad (4)$$

corresponding to the original functional J[y], then v = v(u) satisfies the Euler's equation.

$$\frac{\partial F_1}{\partial v} - \frac{d}{du}\left(\frac{\partial F_1}{\partial v'}\right) = 0 \qquad (5)$$

corresponding to the new functional $J_1[v]$. To prove this, we use the concept of variational derivative.

Let $\Delta\sigma$ denotes the area bounded by the curves y = y(x) and y = y(x) + h(x) and let $\Delta\sigma_1$ denotes the area bounded by corresponding curves v = v(u) and v = v(u) + η(u) in the uv – plane.

Now (by a standard result for the transformation of areas) as $\Delta\sigma$ and $\Delta\sigma_1$ tend to zero, the ratio $\Delta\sigma/\Delta\sigma_1$ approaches the Jacobian $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$ which is non zero by (2).

Thus if $\lim\limits_{\Delta\sigma\to 0} \dfrac{J[y+h] - J[y]}{\Delta\sigma} = 0$, then, $\lim\limits_{\Delta\sigma\to 0} \dfrac{J[v+\eta] - J_1[v]}{\Delta\sigma_1} = 0$

It follows that if y(x) satisfies (4), then v(x) satisfies (5). This proves invariance of Euler's equation on changing the coordinate system.

Remark. In solving Euler's equation sometimes change of variables can be used for simplicity. Because of the invariance property, the change of variables can be made directly in the integral rather than in Euler;s equation and we can then write Euler's equation for new integral.

4.7.3. Example. Find the extemals of the functional $\int_{\theta_1}^{\theta_2} \sqrt{r^2 + r'^2}\, d\theta$ where r = r(θ)

Solution. Let $J[r(\theta)] = \int_{\theta_1}^{\theta_2} \sqrt{(r^2 + r'^2)}\, d\theta$

Put $x = r\cos\theta$, $y = r\sin\theta \Rightarrow \dfrac{dx}{d\theta} = -r\sin\theta + \dfrac{dr}{d\theta}\cos\theta$

and $\dfrac{dy}{d\theta} = r\cos\theta + \dfrac{dr}{d\theta}\sin\theta$

Squaring and adding, we get,

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = r^2 + r'^2$$

Thus, we have

$$\int \sqrt{r^2 + r'^2}\, d\theta = \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int \sqrt{(dx)^2 + (dy)^2}$$

$$= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int \sqrt{1 + y'^2}\, dx$$

Suppose at $\theta = \theta_1$, $x = x_1$ and at $\theta = \theta_2$, $x = x_2$ so that the given functional becomes,

$$J[y(x)] = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

which has been solved earlier and gives the solution of the type

$$y = mx + c$$

Thus, the extremals are $r \sin\theta = m\, r\, x\!\cos\theta + c$

4.7.4. Exercise. Find the extremal of the functional $\int_{\theta_1}^{\theta_2} r \sin\theta \sqrt{r^2 + r'^2}\, d\theta$ using the transformation $x = r\cos\theta$, $y = r\sin\theta$.

4.8. The fixed end point problem for n unknown functions.

4.8.1. Theorem. A necessary condition for the curve $y_i = y_i(x)$ $(i = 1, 2, \ldots, n)$ to be an extremal of the functional

$$J[y_1, y_2, \ldots, y_n] = \int_a^b F(x_1, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')\, dx$$

is that the functions $y_i(x)$ satisfy the Euler's equations.

$$F_{y_i} - \frac{d}{dx}\left(F_{y_i'}\right) = 0 \qquad (i = 1, 2, \ldots, n)$$

Proof : Let $F(x_1, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')$ be a function with continuous first and second derivatives w.r.t. all its arguments. Consider the functional.

$$J[y_1, y_2, \ldots, y_n] = \int_a^b F(x_1, y_1, \ldots, y_n, y_1', \ldots, y_n')\, dx \qquad (1)$$

satisfying the boundary conditions,

$y_i(a) = A_i,\ y_i(b) = B_i\ (i = 1, 2, \ldots, n)$ (2)

We replace each $y_i(x)$ by a varied function $y_i(x) + h_i(x)$ where both $y_i(x)$ and $y_i(x) + h_i(x)$ satisfy the boundary conditions (2). For this, we must have,

$h_i(a) = h_i(b) = 0$ $(i = 1, 2, \ldots, n)$

We now calculate the increment

$$\Delta J = J[y_1 + h_1, \ldots, y_n + h_n] - J[y_1, \ldots, y_n]$$

$$\Rightarrow \Delta J = \int_a^b F(x_1, y_1 + h_1, \ldots, y_n + h_n + y_1' + h_1', \ldots, y_n' + h_n') - F(x_1, y_1, \ldots, y_n, y_1', \ldots, y_n')\,dx$$

Using Taylor's theorem, $\Delta J = \int_a^b \sum_{i=1}^{n} \left(F_{y_i} h_i + F_{y_i'} h_i'\right) dx + \ldots$

The integral on R.H.S. represents the principal linear part of ΔJ and hence the variation of $J[y_1, \ldots, y_n]$ is

$$\delta J = \int_a^b \sum_{i=1}^{n} \left(F_{y_i} h_i + F_{y_i'} h_i'\right) dx$$

Since all the increments $h_i(x)$ are independent, we can choose one of them arbitrarily setting all others equal to zero, so that the necessary condition $\delta J = 0$ for an extremum implies

$$\int_a^b \left(F_{y_i} h_i + F_{y_i'} h_i'\right) dx = 0 \qquad (i = 1, 2, \ldots, n)$$

Using lemma (4) (earlier proved), we obtain

$$F_{y_i} = \frac{d}{dx}\left(F_{y_i}\right) \quad \text{or} \quad F_{y_i} - \frac{d}{dx} F_{y'_i} = 0 \qquad (i = 1, 2, \ldots, n)$$

which are required Euler's equations. This is a system of n second order differential equations, its general solution contains 2n arbitrary constants, which are determined from the boundary conditions (2).

4.8.2. Example. Find the extremals of the functional $J[y, z] = \int_{x_0}^{x_1} \left(2yz - 2y^2 + y'^2 - z'^2\right) dx$

Solution : Euler's equations are $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ (1)

and $\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) = 0$ (2)

Here $f = 2yz - 2y^2 + y'^2 - z'^2$

which gives $\frac{\partial f}{\partial y} = 2z - 4y$; $\frac{\partial f}{\partial y'} = 2y'$; $\frac{\partial f}{\partial z} = 2y$; $\frac{\partial f}{\partial z'} = -2z'$

Thus equation (1) and (2) reduce to,

$$2z-4y-\frac{d}{dx}\left(2y'\right)=0 \Rightarrow z-2y-\frac{d^2y}{dx^2}=0 \quad (3)$$

and $$2y-\frac{d}{dx}\left(-2z'\right)=0 \Rightarrow y+\frac{d^2z}{dx^2}=0 \quad (4)$$

From (3), we have $z = 2y + \frac{d^2y}{dx^2} \Rightarrow \frac{d^2z}{dx^2} \equiv 2\frac{d^2y}{dx^2}+\frac{d^4y}{dx^4}$

Putting in (4), we get, $y + 2\frac{d^2y}{dx^2}+\frac{d^4y}{dx^4}=0 \Rightarrow (D^4+2D^2+1)\,y=0$

Aux. Equation is $m^4 + 2m^2 + 1 = 0 \Rightarrow (m^2+1)^2 = 0 \Rightarrow m = \pm i, \pm i$

Hence, the solution is

$$y = (Ax+B)\cos x + (Cx+D)\sin x \quad (5)$$

and z can be obtained by using the relation, $z = 2y + \frac{d^2y}{dx^2}$

which comes out to be, $z = (Ax+B)\cos x + (Cx+D)\sin x + 2C\cos x - 2A\sin x$ (6)

Equations (5) and (6) are required solutions where A, B, C, D can be determined by the boundary conditions.

4.8.3. Exercise. Find the extremals of the functionals

$J[x_1, x_2] = \int_0^{\pi/2}\left(x_1'^2+x_2'^2+2x_1x_2\right)dt$ subject to boundary conditions

$x_1(0) = 0,\ x_1(\pi/2) = 1,\ x_2(0) = 0,\ x_2(\pi/2) = -1$

Answer. $x_1(t) = \sin t,\ x_2(t) = -\sin t$

4.9. Check Your Progress.

1. Find the extremals of the functional $\int_0^{\pi}\left[y'^2-y^2+uy\cos x\right]dx$, given that $y(0) = 0 = y(1)$.

Answer. $y = (B + x)\sin x$ where B is an arbitrary constant.

4.10. Summary. In this chapter, we observed that to find maxima and minima of functional small changes in functions and functionals were made to derive the required equations and hence some ordinary / partial differential equations were obtained, solving which we achieved the functions which result the functional in extreme values.

Books Suggested:

1. Hilderbrand, F.B., Methods of Applied Mathematics, Dover Publications.
2. Gelfand, J.M., Fomin, S.V., Calculus of Variations, Prentice Hall, New Jersey, 1963.